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DOMINATING SETS OF UNITARY DIVISOR CAYLEY GRAPHS

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ABSTRACT

Let n ≥ 1 be an integer and S be the set of unitary divisors of n.Then the set S*= {s, n- s / s∈ S, n ≠ s} is a symmetric subset of the group $(Zn, \oplus)$, the additive abelian group of integers modulo n. The Cayley graph of (Zn, Un), associated with the above symmetric subset S* is called the Unitary Divisor Cayley graph and it is denoted by G(Zn, Un) .That is, G(Zn, Un) is the graph whose vertex set is V= {0,1,2,…, n-1} and the edge set is E = {(x, y) / x-y or y-x is in S*}. Let G (V, E) be a graph. A subset D of V is said to be a dominating set of G if every vertex in V \ D is adjacent to a vertex in D. A dominating set with minimum cardinality is called a minimum dominating set and its cardinality is called the domination number of G and is denoted by ϒ (G).

KEYWORDS: Unitary Divisor Cayley Graph, Dominating Set, Domination Number

1. INTRODUCTION

Berge [4] and Ore [12] are the first to introduce the concept of domination and they have contributed significantly to the theory of domination in graphs. E. J. Cockayne and S.T. Hedetniemi [6] published the first paper entitled "optimal domination in graphs". They were the first to use the notation $\gamma(G)$ for the domination number of a graph, which subsequently became the accepted notation. Allan and Laskar [1], Cockayne and Hedetniemi [6], Arumugam [3], Sam path Kumar [13] and others have contributed significantly to the theory of dominating sets and domination numbers. An introduction and an extensive overview on domination in graphs and related topics are given by Haynes et al. [8]. In the sequel edited by Haynes, Hedetniemi and Slater [9], several authors presented a survey of articles in the wide field of domination in graphs. Cockayne, C.J et.al [5] introduced the concept of total domination in graphs and studied extensively. The applications of both domination and total domination are widely used in Networks.

In this chapter, we discuss dominating sets and total dominating sets of Unitary Divisor Cayley Graphs. Here we have presented an algorithm which finds all minimal dominating and total dominating set of all n. This algorithm finds all closed and open neighborhood sets respectively, and then finds a minimum number of these set which cover all the vertices of G (Z_n, Un). Each theorem is strengthened with examples and the Algorithm is also illustrated for different values of n.

2. DOMINATING SETS OF UNITARY DIVISOR CAYLEY GRAPH

Let us define a dominating set as follows

Definition

Let G (V, E) be a graph. A subset D of V is said to be a dominating set of G if every vertex in V \ D is adjacent to a vertex in D.

A dominating set with minimum cardinality is called a minimum dominating set and its cardinality is called the domination number of G and is denoted by $\gamma(G)$.

We now find dominating sets of Unitary Divisor Cayley graphs.

Theorem 2.1

If n is a prime or a power of a prime, then D = {rd0 / 0 δ r δ k-1, where k is the largest positive integer such that $rd_0 < n$ and $d_0 = 3$} is a dominating set of G (Z_n, U_n).

Proof

Let n be a prime or a power of a prime.

Then G (Z_n, U_n) is an outer Hamilton cycle.

Let D = {rd0 / 0 δ r δ k-1, where k is the largest positive integer such that $rd_0 < n$ and $d_0 = 3$}.

If we construct D, the vertices are in the form {0, 3, 6, 9,….} and each vertex in D is adjacent with its preceding and succeeding vertices i.e., every vertex in V-D is adjacent with atleast one vertex in D.

Therefore, D becomes a dominating set.

Also, by the construction of D, it is clear that D is minimal since no proper subset of D is a dominating set and hence |D| = k is the domination number of G (Z_n, U_n).

Example 2.2

Consider G (Z_7, U_7) for n = 7 which is a prime.

Then S = {1, 7} and S^* = {1, 6}.

Further the graph G (Z_7, U_7) is 2 – regular.

The graph of G (Z_7, U_7) is given below.

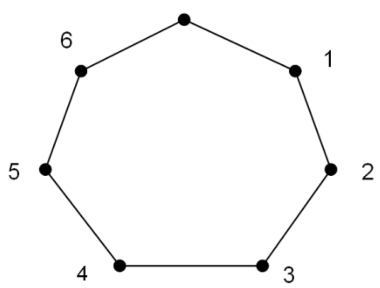

Figure 1: G (Z_7, U_7)

Also, G (Z_7, U_7) is an Outer Hamilton Cycle.

If we construct D, we get D = {0, 3, 6}.

We observe that the set D = {0, 3, 6} dominates all the vertices in V – D.

Further, this D is a minimal dominating set because if we delete one vertex from this site, then the remaining vertices cannot dominate the vertices of G (Z_7, U_7).

Therefore, the domination number of G ($Z_{7,}$ U_7) is |D| = 3.

A possible number of MDSs of G ($Z_{7,}$ U_7) are 7 and these sets are given by

{0,3,6}, {1,4,0}, {2,5,1}, {3,6,2}, {4,0,3}, {5,1,4}, {6,2,5}.

Example 2.3

Consider G ($Z_{8,}$ U_8) for n = 8 which is a power of a prime.

Then S = {1, 8} and S^* = {1,7}.

The graph G ($Z_{8,}$ U_8) is 2 – regular.

The graph of G ($Z_{8,}$ U_8) is given below.

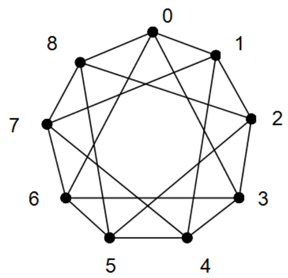

Figure 2: G ($Z_{8,}$ U_8)

Also, G ($Z_{8,}$ U_8) is an outer Hamilton cycle.

If we construct D, we get D = {0, 3, 6}.

We observe that the set D = {0, 3, 6} dominates all the vertices in V – D, since the graph is an outer Hamilton cycle.

Further, this D is a minimal dominating set because if we delete one vertex from this set then the remaining vertices cannot dominate the vertices in G ($Z_{8,}$ U_8).

Therefore, the domination number of G ($Z_{8,}$ U_8) is |D| = 3.

A possible number of MDSs of G ($Z_{8,}$ U_8) are 8 and these sets are given by

{0,3,6}, {1,4,7}, {2,5,0}, {3,6,1}, {4,7,2}, {5,0,3}, {6,1,4}, {7,2,5}.

Remark

The domination number of G (Z_n, U_n) is 1 if n = 2,3and 6. The symmetric subset S^* contains all the vertices of G (Z_n, U_n) for all value of n and hence the difference of any two vertices is in S^*. Thus, every vertex is adjacent to all the vertices of G (Z_n, U_n) so that it is complete. So, every single time vertex is dominating set and hence the domination number is 1.

Theorem 2.4

Let n be not a prime or power of a prime. If n = p_1 .p_2 where p_1 and p_2 are prime, then D = {rd0 / 0 δ r δ k-1, where k is the largest positive integer such that rd_0 < n and d_0 = 5} is a dominating set of G(Z_n, U_n).

Proof

Let n be not a prime or power of a prime.

Suppose $n = p_1.p_2$ where p_1 and p_2 are primes.

Consider the graph G (Z_n, U_n).

In this case, S= $\{1, p_1, p_2, p_1. p_2\}$ and $S^*= \{1, p_1, p_2, p_1. p_2\text{-}1, p_1p_2 - p_2, p_1p_2\text{-}p_1\}$

The graph is $|S^*|$ - regular. .

Let D = $\{rd_0 / 0$ δ r δ k-1, where k is the largest positive integer such that $rd_0 < n$ and $d_0 = 5\}$.

We now show that D is a dominating set of G (Z_n, U_n).

Clearly, the vertex 0 is adjacent to the vertices 1, p_1, p_2, $p_1. p_2\text{-}1$, $p_1p_2 - p_2$, $p_1p_2\text{-}p_1$.

The vertex 5 is adjacent to the vertices 6, $p_1+ 5$, $p_2 +5$, $p_1\ p_2+4$, $p_1p_2 - p_2+5$, $p_1p_2\text{-}p_1+5$.

The vertex 10 is adjacent to the vertices 11, $p_1+ 10$, $p_2 +10$, $p_1. p_2+9$, $p_1p_2 - p_2+10$, $p_1p_2 \text{-}p_1+10$

⋮

Here we observe the vertices {0, 5, 10,….} are dominating the vertices 0, 1, 5, 6, 10, 11, …..

If we give the value for p_1 and p2, then the rest of the vertices which are dominated by {0, 1, 5, 6, 10, 11, …..} will also be found and these vertices are the entire vertices of the vertex set of G (Z_n, U_n).

Likewise, the vertices in D dominate the vertices of the graph G (Z_n, U_n) and hence D is a dominating set of G (Z_n, U_n).

Example 2.5

Consider G (Z_{10}, U_{10}) for n = 10.

Then S = {1, 2, 5, 10}, S^* = {1, 2, 5, 8. 9}.

The graph of G (Z_{10}, U_{10}) is given below.

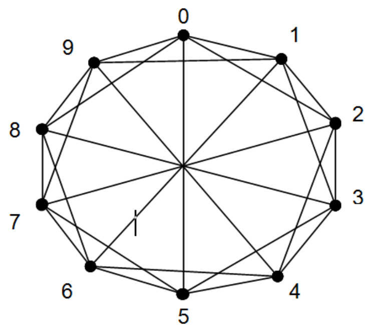

Figure 3: G (Z_{10}, U_{10})

Then D becomes {0, 5} and the above table becomes

0 → 1, 2, 5, 8, 9

5→ 6, 7, 0, 3, 4

Here's the set {0,5} dominates the vertices {0, 1,2, 3, 4, 5, 6, 7, 8, 9}

Hence D = {0,5} becomes a dominating set of G (Z_{10}, U_{10}).

Further D is minimal. If we delete any one vertex from D, then the resulting set will not be a dominating set of G (Z_{10}, U_{10}), since the graph is 5 regular, a set of one vertex cannot dominate all the vertices of the graph G (Z_{10}, U_{10}).

Remark

The dominating set given in Theorem 5.2.4 is not minimal for p_1 = 3, p_2 = 11. This can be seen in the following example.

Example 2.6

Consider G (Z_{33}, U_{33}) for n = 33.

Then S = {1, 3, 11, 33}, S^* = {1, 3, 11, 22, 30, 32}.

The graph of G ($Z_{33,}$ U_{33}) is given below.

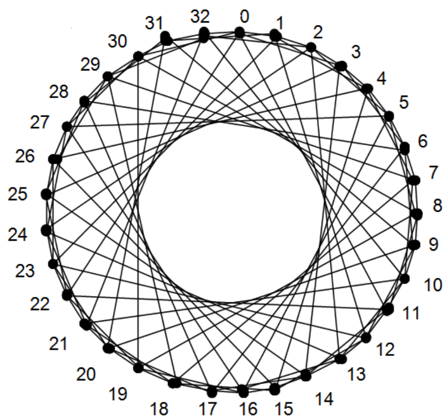

Figure 4: G ($Z_{33,}$ U_{33})

Then D becomes {0, 5, 10, 15, 20, 25, 30} and the above table becomes.

0 → 1, 3, 11, 22, 30, 32

5 → 6, 8, 16, 27, 2, 4

10 → 11, 13, 21, 32, 7, 9

15 → 16, 18, 26, 4, 12, 14

20 → 21, 23, 31, 9, 17, 19

25 → 26, 28, 4, 14, 22, 24

30 → 32, 34, 9, 19, 27, 29

Here D = {0, 5, 10, 15, 20, 25, 30} dominates all the vertices of G (Z_{33}, U_{33}).

Hence D becomes a dominating set of G (Z_{33}, U_{33}).

Since G (Z33, U_{33}) is 6- regular, we should have |D| (6.

Here we have obtained |D| = 7 i.e., D is not minimal.

To get minimal dominating sets, we give the following Algorithm which finds the minimal dominating sets of G (Z_n, U_n).

If we run the Algorithm, then the minimal dominating set obtained from G (Z_{33}, U $_{33}$) is {0, 5, 18, 9, 25, 12} whose cardinality is 6.

The following Algorithm finds minimal dominating sets of G(Z_n, U_n) for all values of n except when n is a prime or a power of a prime. Because when n is a prime, the graph becomes an outer Hamilton cycle and hence the dominating sets are found easily

Algorithm – DS - UDCG

INPUT: Enter a number

OUTPUT: Minimal dominating sets

STEP 1: Enter a number n

STEP 2: IF n IS NOT EQUAL TO NULL

- GOTO STEP 3
- ELSE GOTO STEP 12

STEP 3: METHOD setofdivisiors ($number)

- INITIALIZE ARRAY $divisorset
- FOR EACH $i, $i=1,...... $i<=$number
- DO THE FOLLOWING
- INITIALIZE $modval
- ASSIGN $number % $i TO $modval
- IF $modval EQUAL TO 0 ASSIGN $i TO $divisorset []
- END IF
- RETURN $divisorset
- END FOR

STEP 4: METHOD gcd (param $n, param $m)

- IF ($n EQUAL TO 0 AND $m EQUAL TO 0)
- RETURN 1; //avoid infinite recursion
- END IF
- IF ($n EQUAL TO $m AND $n GREATER THAN EQUAL TO 1)
- RETURN $n;
- END IF
- RETURN $m LESS THAN $n IF gcd ($n-$m, $n) ELSE gcd ($n, $m-$n)

STEP 5: Find sets of divisors

- CALL METHOD set of divisiors(param)
- INITIALIZE VARIABLE $result ASSIGN COUNT OF (set of divisiors (param))
- INITIALIZE VARIABLE $set Of Divisors Array ASSIGN set of divisions (param)
- PRINT "Set of divisors = {";
- FOR EACH $i, $i=0,...... $i<param
- INITIALIZE ARRAY $set Of Divisors Array Temp []
- ASSIGN $set Of Divisors Array [$i];
- PRINT $set Of Divisors Array [$i];
- IF $i NOT EQUAL TO $result-1
- PRINT ",";
- END IF
- END FOR
- PRINT "}"

STEP 6: Find Set of unitary divisors

- INITIALIZEVARIABLE $set Of Divisors Count ASSIGN COUNT OF ($set Of Divisors Array Temp);
- INITIALIZE ARRAY $first Temp Array
- PRINT Set of unitary divisors= {"
- FOR EACH $i, $i=0,...... $i<$set Of Divisors Count
- IF ($i LESS THAN $setOfDivisorsCount-1)
- INITIALIZE VARIABLE $n
- ASSIGN VALUE $i+1 TO $n

- IF (gcd ($first Temp Array [$i], ($number/$first Temp Array [$i])) EQUALTO 1)
- PRINT $first Temp Array [$i]
- ASSIGN VALUE $first Temp Array [$i] TO $gcdarray []
- PRINT ","
- END IF
- END IF
- IF ($i EQUAL TO $setOfDivisorsCount-1)
- IF (gcd ($first Temp Array [$i], $first Temp Array [0]) EQUAL TO 1)
- PRINT $first Temp Array [$i];
- ASSIGN VALUE $first Temp Array [$i] TO $gcdarray []
- END IF
- END IF
- END FOR
- PRINT "}"

STEP 7: Find Symmetric sub set

- INITIALIZEVARIABLE $set Of Divisors Count ASSIGNCOUN OF ($set Of Divisors Array Temp);
- INITIALIZE ARRAY $first Temp Array
- INITIALIZE ARRAY $second Temp Array
- FOR EACH $j, $j=0,.......... $j<$set Of Divisors Count
- IF $set Of Divisors Array Temp [$j] NOT EQUAL TO $number
- IF $set Of Divisors Array Temp [$j] EQUAL TO $number
- BREAK
- ELSE
- ASSIGN VALUE $setOfDivisorsArrayTemp [$j] TO $firstTempArray []
- ASSIGN VALUE ($number-$setOfDivisorsArrayTemp [$j]) TO $secondTempArray []
- END IF
- END IF
- END FOR

- MERGE $first Temp Array AND second Temp Array array_ merge ($first Temp Array, $second Temp Array)
- INITIALIZE VARIABLE $merge FSTemps
- REMOVE DUPLICATE VALUES USING
- array_ unique (array_ merge ($first Temp Array, $second Temp Array))
- ASSIGN TO $mergeFSTemps
- SORT ARRAY asort($mergeFSTemps);
- INITIALIZE VARIABLE $keynum
- ASSIGN 0 TO $keynum
- WHILE (list ($key, $val) = EACH ($mergeFSTemps)) {
- INITIALIZE ARRAY $symmetricSubSet
- ASSIGN VALUE $val TO $symmetricSubSet []
- INCREMENT $keynum++;
- END WHILE
- INITIALIZE VARIABLE $symmetric Sub Set Count
- ASSIGN COUNT OF count ($symmetric Sub Set) TO $symmetric Sub Set Count
- PRINT "Symmeric subset = {"
- FOR EACH $m, $m=0,.......... $m<$symmetric Sub Set Count
- PRINT $symmetric Sub Set [$m];
- IF $m NOT EQUAL TO $symmetricSubSetCount-1
- PRINT ",";
- END IF
- END FOR
- PRINT "}"

STEP 8: Find Neighbourhood sets of n

- FOR EACH $nsa, $nsa=0,.......... $nsa<$number
- FOR EACH $nsb, $nsb=0,.......... $nsb<$symmetric Sub Set
- INITIALIZE MULTI DIMENSIONAL ARRAY $neighbourhood Set Array
- ASSIGN $symmetric Sub Set [$nsb] TO $neighbourhood Set Array [$nsa] [$nsb]

- ASSIGN $symmetric Sub Set [$nsb] +1 TO $symmetric Sub Set [$nsb]
- IF $symmetric Sub Set [$nsb] EQUAL TO $number
- ASSIGN 0 TO $symmetric Sub Set [$nsb]
- END IF
- END FOR
- END FOR
- //FIRST ELEMENTS UNION
- FOR EACH $ns, $ns=0,........... $ns<$number
- PUSH TO FIRST PLACE array_ un shift ($neighbourhood Set Array [$ns], $ns)
- END FOR
- PRINT "Neighbourhood sets of n are"
- FOR EACH $ns, $ns=0,........... $ns<$number
- PRINT "N [$ns] = {"
- FOR EACH $nbs, $nbs=0,........... $nbs <count ($neighbourhood Set Array [$ns]
- PRINT $neighbourhood Set Array [$ns] [$nbs];
- IF $nbs EQUAL TO count ($neighbourhood Set Array [$ns]) -1
- PRINT ""
- ELSE
- PRINT ","
- END FOR
- PRINT "}";
- END FOR

STEP 9: PRINT "Consider N [0] = {";

- FOR EACH $m, $m=0,.......... $m<count ($neighbourhood Set Array [0])
- INITIALIZE ARRAY $nof0array
- ASSIGN $neighbourhood Set Array [0] [$m] TO $nof0array []
- PRINT $neighbourhood Set Array [0] [$m];
- IF $m NOT EQUAL TO count ($neighbourhood Set Array [0]) -1

- PRINT ",";
- END IF
- END FOR
- PRINT "}"

STEP 10: Find Uncovered vertices in N [0]

- FOR EACH $m, $m=0,.......... $m<$number
- INITIALIZE ARRAY $notin
- IF!in_array ($m, $nof0array)
- ASSIGN$m TO $notin []
- END IF
- END FOR
- PRINT "Uncovered vertices in N [0] are {";
- FOR EACH $m, $m=0,.......... $m<count ($notin)
- PRINT $notin [$m];
- IF $m NOT EQUAL TO count ($notin) -1)
- PRINT ",";
- END IF
- END FOR
- PRINT "}";

STEP 11: Find Minimal dominating set

- INITIALIZE $nofzerocount
- ASSIGN count ($neighbourhoodSetArray [0]) TO $nofzerocount
- INITIALIZE $after_merge
- ASSIGN $neighbourhoodSetArray [0] TO $after_merge
- INITIALIZE VARIABLE $nsstart=1;
- INITIALIZE ARRAY $selected_neigh_sets
- REFERENCE OF GO TO a:
- FOR EACH $nss, $nss=$nsstart,.......... $nss<count ($neighbourhoodSetArray)
- IF NOT IN ARRAY ($nss, $selected_neigh_sets) AND count ($notin) NOT

- EQUAL TO NULL
- INITIALIZE VARIABLE $result
- ASSIGN array_intersect ($neighbourhood Set Array [$nss], $notin) TO $result
- IF count ($result) EQUAL TO $nofzerocount
- MERGE $after_merge, $neighbourhoodSetArray [$nss] AND ASSIGN
- TO $after_merge
- ASSIGN array_diff ($notin, $after_ merge) TO $notin
- ASSIGN $nss TO $selected_ neigh_ sets []
- ELSE
- IF count ($neighbourhood Set Array) -1) EQUAL TO $nss
- ASSIGN $nsstart=1;
- ASSIGN $nofzerocount=$nofzerocount-1;
- GOTO a;
- END IF
- END IF
- END IF
- END FOR
- PRINT "Minimal dominating set is = {";
- FOR EACH $m, $m=0,.......... $m<count ($selected_neigh_sets)
- PRINT $selected_neigh_sets [$m];
- IF $m NOT EQUAL TO count ($selected_neigh_sets) -1)
- PRINT ",";
- END IF
- END FOR
- PRINT "}";

STEP 12: FIND the number of minimal dominating sets

- INITIALIZE $finalone
- ASSIGN $selected_neigh_sets TO $finalone

- PRINT "The number of minimal dominating sets of n is"
- FOR EACH $nsone, $nsone=0,.......... $nsone<$number
- PRINT "{";
- FOR EACH $nbsone, $nbsone=0,.......... $nbsone<count ($finalone)
- IF $finalone [$nbsone] EQUAL TO $number $finalone [$nbsone] =0;
- END IF
- PRINT $finalone [$nbsone]
- IF $nbsone EQUAL TO (count ($finalone) -1)
- PRINT "";
- ELSE
- PRINT ",";
- END IF
- INCREMENT $finalone [$nbsone] ++
- END FOR
- PRINT "}";
- INITIALIZE ARRAY $tempfinalone
- ASSIGN $finalone TO $tempfinalone []
- SORT $tempfinalone [$nsone]
- SORT $selected_ neigh_ sets
- IF count (array_ diff ($temp final one [$ns one], $selected_ neigh_ sets) EQUAL
- TO 0
- BREAK;
- ELSE
- PRINT ",";
- Continue;
- END IF
- END FOR

STEP 13: PRINT "The domination number is ". counts ($selected_neigh_sets)

STEP 14: PRINT "please enter a proper number"

- STOP

Theorem 2.7

Let n be not a prime or power of a prime. If n ≠ $p_1.p_2$ where p_1 and p_2 are primes and if n = p_1^m . p_2 where p_1 and p_2 are primes such that m ≥ 2 and $p_2 > 3$ is also a prime then

D= {rd_0 / 0 (r ≤ k-1, where k is the largest positive integer such that rd_0 <n, and d_0 =3} is a dominating set of G (Z_n, U_n). Otherwise, D= {rd0 / 0 ≤ r ≤ k-1, where k is the largest positive integer such that rd_0 < n, and d_0 =6} is a dominating set of G (Z_n, U_n).

Proof

Let n be not a prime, power of a prime or n ≠ $p_1.p_2$.

Case 1

Let n = p_1^m . p_2 where p_1 and p_2 are primes such that m ≥ 2 and p_2 > 3 is also a prime.

Consider the graph G (Z_n, U_n).

In this case, S = {1, p_1, p_1^2 …., $p_1^m, p_2, p_1.p_2, p_1^2.p_2, \ldots, p_1^m.p_2$}

S^*={1,p_1, $p_1^2, \ldots, p_1^m, p_2, p_1.p_2, p_1^2.p_2, \ldots, p_1^m.p_2, p_1^m.p_2-1, n-p_1^m.p_2, \ldots$

$n-p_1^2.p_2, n-p_1.p_2, n-p_2, n-p_1^m, \ldots, n-p_1^2, n-p_1$}.

The graph is $|S^*|$ - regular.

Let D = { rd_0 / 0 ≤ r ≤ k-1 where k is the largest positive integer such that rd_0 < n and d_0 = 3}

i.e., D = { 0, 3, 6, 9,….}.

We now show that D is a dominating set of G(Z_n, U_n).

The set of vertices which are adjacent with the vertices of D is given in the following.

$0 \to 1$, p_1, $p_1^2, \ldots, p_1^m, p_2, p_1.p_2, p_1^2-p_2, \ldots, p_1^m.p_2, p_1^m.p_2-1, n-p_1^m.p_2,$

$\ldots, n-p_1^2.p_2, n-p_1.p_2, n-p_2, n-p_1^m, \ldots, n-p_1^2, n-p_1.$

$3 \to 4$, p_1+3, $p_1^2+3, \ldots, p_1^m+3, p_2+3, p_1.p_2+3, p_1^2.p_2+3, \ldots, p_1^m.p_2+3,$

$p_1^m-p_2+2, n-p_1^m.p_2+3, \ldots, n-p_1.p_2+3, n-p_1.p_2+3, n-p_2+3,$

$n-p_1^m+3, \ldots, n-p_1^2+3, n-p_1+3.$

$6 \to$ **7,** p_1+6, $p_1^2+6, \ldots, p_1^m+6, p_2+6, p_1.p_2+6, p_1^2.p_2+6, \ldots,$

$$p_1^m - p_2 + 6,\ p_1^m.p_2 + 5,\ n - p_1^m.p_2 + 6,\ ....,\ n - p_1^2.p_2 + 6,\ n - p_1.p_2 + 6,$$

$$n - p_2 + 6,\ n - p_1^m + 6,\ ....,\ n - p_1^2 + 6,\ n - p_1 + 6.$$

$$\vdots$$

Here we observe that the vertices {0, 3, 6,….} are dominating the vertices 1,4,7,…..

If we give values for p_1, p_2 and m, then we can see that the vertices of $G(Z_n, U_n)$ are dominated by {0, 3, 6,….} will also be found.

Thus D becomes a dominating set of $G(Z_n, U_n)$.

Case 2

Let $n \neq p_1^m.p_2$ where p_1 and p_2 are primes such that $m \geq 2$ and $p_2 > 3$.

Let $n = p_1^{\alpha 1},\ p_2^{\alpha 2}\ .....\ p_m^{\alpha m}$.

In this case, $S = \{1, p_1, ..., p_1^{\alpha 1}\ ...., p_2, p_2^2 ......\ p_2^{\alpha 2}, p_3, p_3^2 ...., p_3^{\alpha 3}, ...,$

$p_m, p_m^2 ....., p_m^{\alpha m}, p_1 . p_2, p_1 . p_2^2, p_1 . p_2^3 ....., p_1 . p_2^{\alpha 2},$

$p_1^2 . p_2, p_1^3 . p_2, ..., p_1^{\alpha 1} . p_2, p_2 . p_3, p_2 . p_3^2, ..., p_2 . p_3^{\alpha 3},$

$........, p_{m-1} . p_m, ..., p_{m-1} . p_m^{\alpha m}, ...\ p^2{}_{m-1} . p_m, p^3{}_{m-1} . p_m,$

$p_{m-1}^{\alpha m-1} . p_m, p_1^{\alpha 1}\ p_2^{\alpha 2} .....p_m\}$

$S^* = \{1, p_1, ..., p_1^{\alpha 1}\ ...., p_2, p_2^2 ......\ p_2^{\alpha 2}, p_3, p_3^2 ...., p_3^{\alpha 3}, ...,$

$p_m, p_m^2 ....., p_m^{\alpha m}, p_1 . p_2, p_1 . p_2^2, p_1 . p_2^3 ....., p_1 . p_2^{\alpha 2},$

$p_1^2 . p_2, p_1^3 . p_2, ..., p_1^{\alpha 1} . p_2, p_2 . p_3, p_2 . p_3^2, ...,$

$p_2 . p_3^{\alpha 3}, ..., p_{m-1} . p_m, ..., p_{m-1} . p_m^{\alpha m}, ...\ p^2{}_{m-1} . p_m,$

$p^3{}_{m-1} . p_m, p_{m-1}^{\alpha m-1} . p_m, n - 1, n - p_1, ..., n - p_1^{\alpha 1}\ ...., n - p_2,$

$n - p_2^2 ......n - p_2^{\alpha 2}, n - p_3, n - p_3^2 ...., n - p_3^{\alpha 3}, ..., n - p_m . p_m^2 .....,$

n- $p_m^{\alpha m}$, n-p_1. p_2, n- p_1. p_2^2, n- p_1. p_2^3 …..,n- p_1. $p_2^{\alpha 2}$,

n- p_1^2 . p_2, n- p_1^3 . p_2 , ….., n- $p_1^{\alpha 1}$. p_2, n-p_2 . p_3, n- p_2 . p_3^2,

…., n- p_2 . $p_3^{\alpha 3}$, ……..,n- p_{m-1} . p_m, ……,n-p_{m-1}. $p_m^{\alpha m}$, …

n- $p^2{}_{m-1.}$ p_m , n-$p^3{}_{m-1.}$ p_m , n- $p_{m-1}^{\alpha m-1}$. $p_{m,}$ }

The graph is | S* | - regular.

Let D = {rd_0 / $0 \leq r \leq k-1$ where k is the largest integer such that $rd_0 < n$ and $d_0 = 6$}.

We now show that D is a dominating set of G(Z_n, U_n).

The set of the vertices which are adjacent with the vertices of G(Z_n , U_n) are given by

$0 \rightarrow$ 1, p_1, …….., $p_1^{\alpha 1}$ …., p_2, p_2^2 …… $p_2^{\alpha 2}$, p_3, p_3^2 …., $p_3^{\alpha 3}$,….., p_m,

p_m^2….., $p_m^{\alpha m}$, p_1. p_2, p_1. p_2^2, p_1. p_2^3 ….., p_1. $p_2^{\alpha 2}$, p_1^2 . p_2, p_1^3 . p_2 ,

….., $p_1^{\alpha 1}$. p_2, p_2 . p_3, p_2 . p_3^2, …., p_2 . $p_3^{\alpha 3}$, ….p_{m-1} . p_m, ……..

…,p_{m-1}. $p_m^{\alpha m}$, …. $p^2{}_{m-1.}$ p_m , $p^3{}_{m-1.}$ p_m , $p_{m-1}^{\alpha m-1}$. $p_{m,}$ n – 1,

n-p_1, …….., n- $p_1^{\alpha 1}$ …., n-p_2, n- p_2^2 ……, n- $p_2^{\alpha 2}$,n- p_3, n- p_3^2 ….,

n- $p_3^{\alpha 3}$,….., n- p_m. p_m^2….., n- $p_m^{\alpha m}$, n-p_1. p_2, n- p_1. p_2^2,

n- p_1. p_2^3 …..,n- p_1. $p_2^{\alpha 2}$, n- p_1^2 . p_2, n- p_1^3 . p_2 , ….., n- $p_1^{\alpha 1}$. p_2,

n-p_2 . p_3, n- p_2 . p_3^2, …., n- p_2 . $p_3^{\alpha 3}$, ……..,n- p_{m-1} . p_m, ……,

n-p_{m-1}. $p_m^{\alpha m}$, … .n- $p^2{}_{m-1.}$ p_m , n-$p^3{}_{m-1.}$ p_m , n- $p_{m-1}^{\alpha m-1}$. $p_{m,}$

$6 \rightarrow$ 7, p_{1+6}, …….., $p_1^{\alpha 1}$ +6…., p_2+6, p_2^2 + 6…… $p_2^{\alpha 2}$ +6, p_3 + 6,

p_3^2+6…., $p_3^{\alpha 3}$ + 6,….., p_m + 6, p_m^2 + 6….., $p_m^{\alpha m}$ + 6, p_1. p_2 + 6,

p_1. p_2^2 + 6, p_1. p_2^3 + 6….., p_1. $p_2^{\alpha 2}$ + 6, p_1^2 . p_2 + 6, p_1^3 . p_2 + 6 ,

….., $p_1^{\alpha 1}$. p_2 + 6, p_2 . p_3 + 6, p_2 . p_3^2 + 6, …., p_2 . $p_3^{\alpha 3}$ + 6,

$\ldots p_{m-1} . p_m + 6, \ldots, p_{m-1} . p_m^{\alpha m} + 6, \ldots . p^2_{m-1} . p_m + 6, p^3_{m-1} . p_m +$

$6, p_{m-1}^{\alpha m-1} . p_m + 6, n + 5, n- p_1 + 6, \ldots\ldots, n- p_1^{\alpha 1} + 6\ldots, n- p_2$

$+ 6, n- p_2^2 + 6\ldots\ldots, n- p_2^{\alpha 2} + 6, n- p_3+6, n- p_3^2 + 6\ldots, n- p_3^{\alpha 3} +$

$6,\ldots\ldots, n- p_m . p_m^2 + 6\ldots\ldots, n- p_m^{\alpha m} + 6, n-p_1 . p_2 + 6, n- p_1 .$

$p_2^2 +6, n- p_1 . p_2^3 +6,\ldots\ldots\ldots\ldots, n- p_1 . p_2^{\alpha 2} +6, n- p_1^2 . p_2 + 6,$

$n- p_1^3 . p_2+6 , \ldots\ldots\ldots, n- p_1^{\alpha 1} . p_2 + 6, n-p_2 . p_3+6, n- p_2 . p_3^2+6,$

$\ldots, n- p_2 . p_3^{\alpha 3} + 6, \ldots\ldots, n- p_{m-1} . p_m + 6, \ldots\ldots, n-p_{m-1} . p_m^{\alpha m} +6,$

$\ldots . n- p^2_{m-1} . p_m +6, n-p^3_{m-1} . p_m +6, n- p_{m-1}^{\alpha m-1} . p_m + 6.$

$\vdots$

Here we observe that the vertices {0, 6,…. (are dominating the vertices 0,1,5,7,…

If we give the value for $p_1, p_2, \ldots$, pm and $\alpha_1, \alpha_2, \ldots, \alpha_m$ then the rest of the vertices which are dominated by {0,6,…. (will also be found.

Likewise, the vertices in D dominate the vertices of the graph $G(Z_n, U_n)$ and hence D are a dominating set of G (Z_n, U_n).

Example 2.8

Let n = 20 where $p_1 = 2$, $p_2 = 5$, m = 2.

Here S = {1, 4, 5, 20} S^* = {1, 4, 5, 15, 16, 19}.

The graph of G (Z_{20}, U_{20}) is given below.

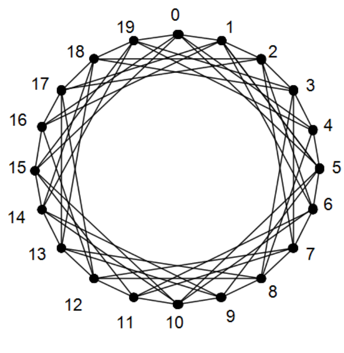

Figure 5: G (Z_{20}, U_{20})

Then D becomes {0, 3, 6, 9, 12} and the vertices adjacent with the vertices of D are

$0 \rightarrow 1, 4, 5, 15, 16, 19$

$3 \rightarrow 4, 7, 8, 18, 19, 2$

$6 \rightarrow 7, 10, 11, 1, 2, 5$

$9 \rightarrow 10, 13, 14, 4, 5, 8$

$12 \rightarrow 13, 16, 17, 7, 8, 11$

i.e., {0, 3, 6, 9, 12} dominates all the vertices of G (Z_{20}, U_{20}).

Hence this set becomes a dominating set of G (Z_{20}, U_{20}).

Since G (Z_{20}, U_{20}) is 6 – regular, we should have $|D| \geq 4$.

Here we obtained $|D| = 5$.

i.e., D is not minimal.

If we run the Algorithm, then the dominating set obtained from this graph is D= {0, 5, 7, 13} whose cardinality is 4 which is minimal.

Example 2.9

Let n = 42

Then S = {1, 2, 3, 6, 7, 14, 21, 42} and S* = {1, 2, 3, 6, 7, 14, 21, 28, 35, 36, 39, 40, 41}.

The graph of G (Z_{42}, U_{42}) is as follows:

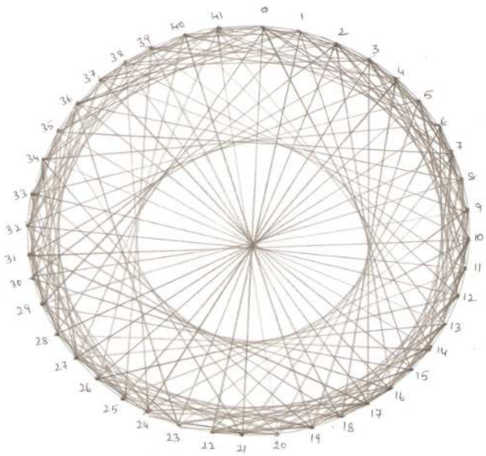

Figure 6: G (Z_{42}, U_{42})

Now D = {0, 6, 12, 18, 24, 30, 36} and the vertices adjacent with the vertices of D are

$0 \rightarrow 1,2,3,6,7,14,21,28,35,36,39,40,41$

$6 \rightarrow 7,8,9,12,13,20,27,34,41,0,3,4,5$

$12 \rightarrow 13,14,15,18,19,26,33,40,5,6,9,10,11$

18 → 19,20,21,24,25,32,39,4,11,12,15,16,17

24 → 25,26,27,30,31,38,3,10,17,18,21,22,23

30 → 31,32,33,36,37,2,9,16,23,24,27,28,29

36 → 37,38,39,0,1,8,15,22,29,30,33,34,35

Therefore, D = {0, 6, 12, 18, 24, 30, 36} dominates all the vertices of G ($Z_{42,}$ U_{42}).

Hence this set becomes a dominating set of G (Z_{42}, U_{42}).

Since G ($Z_{42,}$ U_{42}) is 13-regular, we should have $|D| \geq 4$.

Here we have obtained $|D| = 7$. i.e., D is not minimal.

By the Algorithm, the minimal dominating set of G (Z_{42}, U_{42}) is {0, 10, 26, 16, 6} whose cardinality is 5.

The given algorithm is developed by using the PHP software (server scripting language) and the following illustrations are obtained by simply giving the value for n.

3. ALGORITHM - ILLUSTRATIONS

1. Suppose n = $p_1.p_2$ where p_1 and p_2 are primes.

Let n =33

S = {1, 3, 11, 33}

S* = {1, 3, 11, 22, 30, 32}

Neighbourhood sets of G (Z_{33}, U_{33}) are

N [0] = {0, 1, 3, 11, 22, 30, 32}

N [1] = {1,2,4,12,23,31,0}

N [2] = {2, 3, 5, 13, 24, 32, 1}

N [3] = {3, 4, 6, 14, 25, 0, 2}

N [4] = {4, 5, 7, 15, 26, 1, 3}

N [5] = {5, 6, 8, 16, 27, 2, 4}

N [6] = {6, 7, 9, 17, 28, 3, 5}

N [7] = {7, 8, 10, 18, 29, 4, 6}

N [8] = {8, 9, 11, 19, 30, 5, 7}

N [9] = {9, 10, 12, 20, 31, 6, 8}

N [10] = {10, 11, 13, 21, 32, 7, 9}

N [11] = {11, 12, 14, 22, 0, 8, 10}

N [12] = {12, 13, 15, 23, 1, 9, 11}

N [13] = {13, 14, 16, 24, 2, 10, 12}

N [14] = {14, 15, 17, 25, 3, 11, 13}

N [15] = {15, 16, 18, 26, 4, 12, 14}

N [16] = {16, 17, 19, 27, 5, 13, 15}

N [17] = {17, 18, 20, 28, 6, 14, 16}

N [18] = {18, 19, 21, 29, 7, 15, 17}

N [19] = {19, 20, 22, 30, 8, 16, 18}

N [20] = {20, 21, 23, 31, 9, 17, 19}

N [21] = {21, 22, 24, 32, 10, 18, 20}

N [22] = {22, 23, 25, 0, 11, 19, 21}

N [23] = {23, 24, 26, 1, 12, 20, 22}

N [24] = {24, 25, 27, 2, 13, 21, 23}

N [25] = {25, 26, 28, 3, 14, 22, 24}

N [26] = {26, 27, 29, 4, 15, 23, 25}

N [27] = {27, 28, 30, 5, 16, 24, 26}

N [28] = {28, 29, 31, 6, 17, 25, 27}

N [29] = {29, 30, 32, 7, 18, 26, 28}

N [30] = {30, 31, 0, 8, 19, 27, 29}

N [31] = {31, 32, 1, 9, 20, 28, 30}

N [32] = {32, 0, 2, 10, 21, 29, 31}

The minimal dominating set is = {0, 5, 18, 9, 25, 12}

The minimal dominating sets of G (Z_{33}, U_{33}) are

{0,5,18,9,25,12},{1,6,19,10,26,13},{2,7,20,11,27,14},{3,8,21,12,28,15},{4,9,22,13,29,16},{5,10,23,14,30,17},{6,11,24,15,31,18},{7,12,25,16,32,19},{8,13,26,17,0,20},{9,14,27,18,1,21},{10,15,28,19,2,22},{11,16,29,20,3,23},{12,17,30,21,4,24},{13,18,31,22,5,25},{14,19,32,23,6,26},{15,20,0,24,7,27},{16,21,1,25,8,28},{17,22,2,26,9,29},{18,23,3,27,10,30},{19,24,4,28,11,31},{20,25,5,29,12,32},{21,26,6,30,13,0},{22,27,7,31,14,1},{23,28,8,32,15,2},{24,29,9,0,16,3},{25,30,10,1,17,4},{26,31,11,2,18,5},{27,32,12,3,19,6},{28,0,13,4,20,7},{29,1,14,5,21,8},{30,2,15,6,22,9},{31,3,16,7,23,10},{32,4,17,8,24,11}.

The domination number is 6.

2. Let n = p_1^m . p_2 where p_1 and p_2 are primes such that m ≥2 and $p_2 > 3$ is also a prime. Suppose n =20

S = {1, 4, 5, 20}

S* = {1, 4, 5, 15, 16, 19}

Neighbourhood sets of G (Z_{20}, U_{20}) are

N [0] = {0, 1, 4, 5, 15, 16, 19}

N [1] = {1, 2, 5, 6, 16, 17, 0}

N [2] = {2, 3, 6, 7, 17, 18, 1}

N [3] = {3, 4, 7, 8, 18, 19, 2}

N [4] = {4, 5, 8, 9, 19, 0, 3}

N [5] = {5, 6, 9, 10, 0, 1, 4}

N [6] = {6, 7, 10, 11, 1, 2, 5}

N [7] = {7, 8, 11, 12, 2, 3, 6}

N [8] = {8, 9, 12, 13, 3, 4, 7}

N [9] = {9, 10, 13, 14, 4, 5, 8}

N [10] = {10, 11, 14, 15, 5, 6, 9}

N [11] = {11, 12, 15, 16, 6, 7, 10}

N [12] = {12, 13, 16, 17, 7, 8, 11}

N [13] = {13, 14, 17, 18, 8, 9, 12}

N [14] = {14, 15, 18, 19, 9, 10, 13}

N [15] = {15, 16, 19, 0, 10, 11, 14}

N [16] = {16, 17, 0, 1, 11, 12, 15}

N [17] = {17, 18, 1, 2, 12, 13, 16}

N [18] = {18, 19, 2, 3, 13, 14, 17}

N [19] = {19, 0, 3, 4, 14, 15, 18}

Minimal dominating set is = {0, 7, 13, 5}

The minimal dominating sets of G (Z_{20} , U_{20}) are

{0,7,13,5},{1,8,14,6},{2,9,15,7},{3,10,16,8},{4,11,17,9},{5,12,18,10},{6,13,19,11},{7,14,0,12},{8,15,1,13},{9, 16,2,14},{10,17,3,15},{11,18,4,16},{12,19,5,17},{13,0,6,18},{14,1,7,19},{15,2,8,0},{16,3,9,1},{17,4,10,2}, {18,5,11,3},{19,6,12,4}.

The domination number is 4.

3. Let n = $p_1^{\alpha 1}$, $p_2^{\alpha 2}$ ….. $p_m^{\alpha m}$

Suppose n =42

S = {1, 2, 3, 6, 7, 14, 21, 42}

S^* = {1,2,3,6,7,14,21,28,35,36,39,40,41}

Neighbourhood sets of G (Z_{42}, U_{42}) are

N[0]={0,1,2,3,6,7,14,21,28,35,36,39,40,41}

N[1]={1,2,3,4,7,8,15,22,29,36,37,40,41,0}

N[2]={2,3,4,5,8,9,16,23,30,37,38,41,0,1}

N[3]={3,4,5,6,9,10,17,24,31,38,39,0,1,2}

N[4]={4,5,6,7,10,11,18,25,32,39,40,1,2,3}

N[5]={5,6,7,8,11,12,19,26,33,40,41,2,3,4}

N[6]={6,7,8,9,12,13,20,27,34,41,0,3,4,5}

N[7]={7,8,9,10,13,14,21,28,35,0,1,4,5,6}

N[8]={8,9,10,11,14,15,22,29,36,1,2,5,6,7}

N[9]={9,10,11,12,15,16,23,30,37,2,3,6,7,8}

N[10]={10,11,12,13,16,17,24,31,38,3,4,7,8,9}

N[11]={11,12,13,14,17,18,25,32,39,4,5,8,9,10}

N[12]={12,13,14,15,18,19,26,33,40,5,6,9,10,11}

N[13]={13,14,15,16,19,20,27,34,41,6,7,10,11,12}

N[14]={14,15,16,17,20,21,28,35,0,7,8,11,12,13}

N[15]={15,16,17,18,21,22,29,36,1,8,9,12,13,14}

N[16]={16,17,18,19,22,23,30,37,2,9,10,13,14,15}

N[17]={17,18,19,20,23,24,31,38,3,10,11,14,15,16}

N[18]={18,19,20,21,24,25,32,39,4,11,12,15,16,17}

N[19]={19,20,21,22,25,26,33,40,5,12,13,16,17,18}

N[20]={20,21,22,23,26,27,34,41,6,13,14,17,18,19}

N[21]={21,22,23,24,27,28,35,0,7,14,15,18,19,20}

N[22]={22,23,24,25,28,29,36,1,8,15,16,19,20,21}

N[23]={23,24,25,26,29,30,37,2,9,16,17,20,21,22}

N[24]={24,25,26,27,30,31,38,3,10,17,18,21,22,23}

N[25]={25,26,27,28,31,32,39,4,11,18,19,22,23,24}

N[26]={26,27,28,29,32,33,40,5,12,19,20,23,24,25}

N[27]={27,28,29,30,33,34,41,6,13,20,21,24,25,26}

N[28]={28,29,30,31,34,35,0,7,14,21,22,25,26,27}

N[29]={29,30,31,32,35,36,1,8,15,22,23,26,27,28}

N[30]={30,31,32,33,36,37,2,9,16,23,24,27,28,29}

N[31]={31,32,33,34,37,38,3,10,17,24,25,28,29,30}

N[32]={32,33,34,35,38,39,4,11,18,25,26,29,30,31}

N[33]={33,34,35,36,39,40,5,12,19,26,27,30,31,32}

N[34]={34,35,36,37,40,41,6,13,20,27,28,31,32,33}

N[35]={35,36,37,38,41,0,7,14,21,28,29,32,33,34}

N[36]={36,37,38,39,0,1,8,15,22,29,30,33,34,35}

N[37]={37,38,39,40,1,2,9,16,23,30,31,34,35,36}

N[38]={38,39,40,41,2,3,10,17,24,31,32,35,36,37}

N[39]={39,40,41,0,3,4,11,18,25,32,33,36,37,38}

N[40]={40,41,0,1,4,5,12,19,26,33,34,37,38,39}

N[41]={41,0,1,2,5,6,13,20,27,34,35,38,39,40}

Minimal dominating set is = {0, 10, 26, 16, 6}.

The minimal dominating sets of G (Z_{42}, U_{42}) are

{0,10,26,16,6},{1,11,27,17,7},{2,12,28,18,8},{3,13,29,19,9},{4,14,30,20,10},{5,15,31,21,11},{6,16,32,22,12},{7,17,33,23,13},{8,18,34,24,14},{9,19,35,25,15},{10,20,36,26,16},{11,21,37,27,17},{12,22,38,28,18},{13,23,39,29,19},{14,24,40,30,20},{15,25,41,31,21},{16,26,0,32,22},{17,27,1,33,23},{18,28,2,34,24},{19,29,3,35,25},{20,30,4,36,26},{21,31,5,37,2},{22,32,6,38,28},{23,33,7,39,29},{24,34,8,40,30},{25,35,9,41,31},{26,36,10,0,32},{27,37,11,1,33},{28,38,12,2,34},{29,39,13,3,35},{30,40,14,4,36},{31,41,15,5,37},{32,0,16,6,38},{33,1,17,7,39},{34,2,18,8,40},{35,3,19,9,41},{36,4,20,10,0},{37,5,21,11,1},{38,6,22,12,2}, {39,7,23,13,3}, {40,8,24,14,4}, {41,9,25,15,5}.

CONCLUSIONS

The domination number is 5.

REFERENCES

1. Allan, R.B. and Laskar, R. C. On domination and independent domination numbers of a graph, Discrete Math, 23 (1978), 73-76.
2. Apostol, M. Tom. Introduction to Analytic Number theory, Springer International Student Edition, (1989).
3. Arumugam, S. Uniform Domination in graphs, National Seminar on graph theory and its applications, January (1983).
4. Berge, C. The Theory of Graphs and its Applications, Methuen, London (1962).
5. Cockayne, C.J., Dawes, R.M. and Hedetniemi, S. T. Total domination in graphs, Networks, 10 (1980), 211-219.
6. Cockayne, C.J., and Hedetniemi, S. T. Towards a theory of domination in graphs, Networks, 7 (1977), 247 – 261.
7. Harary, F.Graph Theory, Addison –Wesley, Reading, MA, (1969).
8. Haynes, T.W., Hedetniemi, S.T. and Slater, P. J. Fundamentals of domination in graphs, Marcel Dekker, Inc., New York (1998).
9. Haynes, T.W., Hedetniemi, S.T. and Slater, P. J. Domination in graphs: Advanced Topics, Marcel Dekker, Inc., New York (1998).
10. Madhavi, L. Studies on domination parameters and enumeration of cycles in some Arithmetic Graphs, Ph. D. Thesis, submitted to S.V. University, Tirupati, India, (2002).
11. Nathanson, B. Melvyn Connected components of arithmetic graphs, Monat. fur. Math, 29 (1980), 219 – 220.
12. Ore, O. Theory of Graphs, Amer. Math. Soc. Colloq. Publ. Vol. 38.Amer. Math. Soc., Providence, RI, (1962).
13. Sampath Kumar, E. On some new domination parameters of a graph. A survey. Proceedings of a Symposium on Graph Theory and Combinatorics, Kochi, Kerala, India, 17 – 19 May (1991), 7 – 13.
14. Sujatha, K. Sudies on Domination Parameters and Cycle Structure of Cayley Graphs associated with some Arithmetical Functions, Ph. D. thesis, S. V. University, 2008.
15. Vizing, V. G. Vertex colorings with given colors, Metody Diskret. Analiz. (in Russian) 29:3-10,1976.